

Toy models for wrapping effects

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ABSTRACT: The anomalous dimensions of local single trace gauge invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory can be computed by diagonalizing a long range integrable Hamiltonian by means of a perturbative asymptotic Bethe ansatz. This formalism breaks down when the number of fields of the composite operator is smaller than the range of the Hamiltonian which coincides with the order in perturbation theory at study. We analyze two spin chain toy models which might shed some light on the physics behind these wrapping effects. One of them, the Hubbard model, is known to be closely related to $\mathcal{N} = 4$ SYM. In this example, we find that the knowledge of the effective spin chain description is insufficient to reconstruct the finite size effects of the underlying electron theory. We compute the wrapping corrections for generic states and relate them to a Luscher like approach. The second toy models are long range integrable Hamiltonians built from the standard algebraic Bethe ansatz formalism. This construction is valid for any symmetry group. In particular, for non-compact groups it exhibits an interesting relation between wrapping interactions and transcendentality.

KEYWORDS: Bethe Ansatz, Integrable Field Theories, Lattice Integrable Models, AdS-CFT Correspondence.

Contents

1. Introduction and discussion	1
2. The Hubbard model	7
2.1 The magnon	9
2.1.1 Diagrammatically	11
2.2 Luscher in Hubbard	13
2.2.1 Luscher with magnon-magnon S-matrix	16
2.3 Magnon effective Bethe ansatz equations and generic wrappings	17
2.3.1 Corrected BAE	18
2.3.2 Meeting (and generalizing) Luscher	20
2.3.3 Perturbative treatment	22
3. Families of long-ranged integrable hamiltonians	24
3.1 Generalizations and transcendentality	27
A. Wave function of the Hubbard magnon	30
B. Dressed BDS BAE with twists	32
C. Luscher with magnon-magnon S-matrix at strong coupling	34

1. Introduction and discussion

In [1] the one-loop spectrum of single trace local gauge invariant operators made out of the scalars of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory was reduced to that of a nearest neighbors integrable Hamiltonian with $SO(6)$ symmetry. In particular, single trace operators made out of two complex scalars X and Z were mapped to states in a one dimensional spin 1/2 ring,

$$\text{tr}(ZZ \dots ZXZ \dots ZXZ \dots ZZ) \longleftrightarrow |\uparrow\uparrow \dots \uparrow\downarrow \dots \uparrow\downarrow \dots \uparrow\uparrow\rangle \quad (1.1)$$

Soon after it was understood that integrability persists for the full set of $PSU(2, 2|4)$ fields [2, 3] and at higher orders in perturbation theory [4] where the Hamiltonian becomes long ranged with the range being the order in perturbation theory one considers. Later on, inspired by string theory data [5 – 7], the full $PSU(2, 2|4)$ Bethe equations were proposed [8] and the solutions to these equations are believed to yield the spectrum of generic length L operators up to order g^{2L} . At this order the interactions wrap the single trace operator and invalidate the use of the Bethe ansatz formalism. To achieve such remarkable point

where the spectrum of long operators is believed to be known, a crucial step was required. Namely the idea of looking at operators like (1.1) as a vacuum (the Z fields) on top of which particles (in this example the X fields) propagate [9]. In this language the relevant object becomes the S -matrix scattering these particles, also known as magnons. This S -matrix is $SU(2|2)^2$ extended symmetric [10, 11] and it turns out that symmetry alone almost fixes (up to an overall function) the $(4^4)^2$ entries of this matrix [11–13]. The unfixed overall scalar factor has also been conjectured in [14, 15]. Knowing the S -matrix of the theory it is then possible to write down the Bethe equations quantizing their momenta and, knowing the respective dispersion relation, to compute their energy. For example, states made out of two magnons will be given by

$$\sum_{n_1 \ll n_2} (e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2) e^{ip_1 n_2 + ip_2 n_1}) |n_1, n_2\rangle + \dots \quad (1.2)$$

where $|n_1, n_2\rangle$ represents the state with X fields in the n_1 'th and n_2 'th positions in a sea of $L - 2$ Z fields. The dots correspond to a non-trivial part of the eigenstate in the boundary of the asymptotic region when n_1 is not very far from n_2 and the magnons are strongly interacting. The momenta are then quantized via the Bethe equations

$$e^{ip_1 L} = S(p_1, p_2) \quad , \quad e^{ip_2 L} = S(p_2, p_1)$$

which physically simply mean that the phase acquired by each magnon when going around the ring equals the free propagation phase pL plus the phase shift due to scattering with the other magnons. The spectrum is then given by the sum of energies of the individual magnons as $\Delta - L + 2 = \epsilon_\infty(p_1) + \epsilon_\infty(p_2) + \mathcal{O}(g^{2L})$ where the infinite volume dispersion relation, also fixed by symmetry [11, 16, 13], is given by

$$\epsilon_\infty(p) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}.$$

The simplest possible 2 magnon state is the well known Konishi operator

$$|K\rangle = |\downarrow\uparrow\downarrow\uparrow\rangle - |\downarrow\downarrow\uparrow\uparrow\rangle \quad (1.3)$$

whose dimension can be computed from the known Bethe ansatz equations [4, 8] up to order $g^{2L} = g^8$,

$$\Delta_K = 4 + 12g^2 - 48g^4 + 336g^6 + \mathcal{O}(g^8).$$

At order g^8 wrapping interactions appear and the techniques at hand do not suffice to tackle this computation. Still there are already two possible results in the literature [18, 19] (see also [20]) where the g^8 coefficient was computed by direct evaluation of field theory Feynman diagrams. In figure 1 we plot the several computations, conjectures and speculative guesses, for the Konishi anomalous dimension up to four loops.

Optimistically one might expect to find some extra integrable structure in $\mathcal{N} = 4$ SYM which would allow one to treat the gauge invariant states beyond the perturbative asymptotic Bethe ansatz regime. A particularly appealing possibility would be that some extra hidden local degrees of freedom exist and the long range interactions we perceive

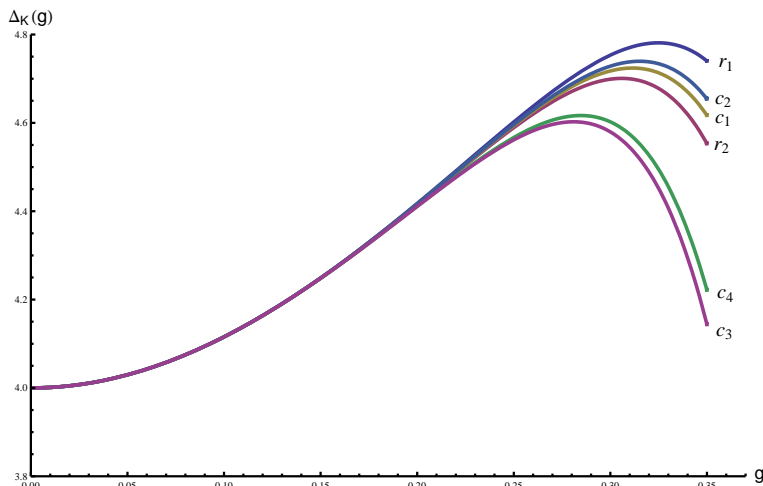


Figure 1: Results and conjectures for the scaling dimension $\Delta(g) = 4 + 12g^2 - 48g^4 + 336g^6 - ag^8$, of the Konishi operator up to four loops, when wrapping interactions first appear. The lines r_1 and r_2 correspond to the two recent (disagreeing) computations in [18, 19] for which $a = 2607 + 28\zeta(3) + 140\zeta(5)$ and $-2584 + 384\zeta(3) - 1440\zeta(5)$ respectively. The first one was done using superspace techniques whereas the latter used component formalism making the comparison between these two laborious computations far from easy. These computations differ from all previous conjectures by the presence of $\zeta(5)$. c_1 is the most recent conjecture [21], which is based on some transcendentality observations and pomeron considerations and predicts $a = -5307/2 + 564\zeta(3)$. Conjectures c_2 and c_3 appear in [14]. The former corresponds to $a = 5640 + 288\zeta(3)$ and would be the value of the anomalous dimension of the Konishi operator if we were to believe the Bethe ansatz equation beyond their natural limit of validity and is therefore a very unlikely proposal [21]. c_3 is the anomalous dimension whose transcendental part is that given by the BAE while the rational part is that predicted by the Hubbard model and has therefore $a = 5088 + 288\zeta(3)$. Finally c_4 with $a = 5088$ would be the result predicted by the Hubbard model [23] which we now know only reproduces the good BAE up to 3 loops.

would rather be the effect of integrating out these fundamental degrees of freedom. This scenario finds compelling evidence at strong coupling in [24–27], at weak coupling in [23] and for general coupling in [28].

In [24–27] quantum sigma models describing the S^n subsector of $AdS_5 \times S^5$ type IIB superstring were seen to reproduce the long range conjectured AFS string Bethe equations [7] at strong coupling when the rapidities (θ 's) of the relativistic particles were integrated out thus leaving an effective Hamiltonian for the isospin degrees of freedom.

In [23] the BDS equations [4], which are known to describe the $SU(2)$ sector of the supersymmetric gauge theory spectrum up to three loops, were shown to be equivalent to the Hubbard model at half filling where again integrating out the momenta (q 's) of the electrons yields an effective long range Hamiltonian with $SU(2)$ symmetry for the spins of the electrons.

In the Hubbard model the effective magnons appearing in (1.2) can be understood as bound states of empty sites (o) and doubly occupied sites (\uparrow). As we will describe

below, if, in the spirit of [29–33], we want to diagrammatically compute the finite size corrections to the effective magnon theory coming from the Hubbard model using the Luscher approach [34, 35], then we need to take into account the fact that the magnon is a bound state of two fundamental particles rather than a fundamental excitation itself. For example, as discussed in section 2.2 the leading finite size corrections to the magnon dispersion relation can be reproduced for *any* value of the coupling g from the expression

$$\delta\epsilon(p) = \sum_{\sigma=o,\uparrow} \left(\frac{1}{2} \text{Res}_q e^{iL(q-\phi_\sigma)} (\epsilon'(q) - \epsilon'_\infty(p)) S_{\hat{\phi},\sigma}^{\hat{\phi},\sigma}(p, q) + c.c. \right) \quad (1.4)$$

where, in order to reproduce the correct result, we must use the scattering matrix between a magnon bound state $\hat{\phi}$ and its fundamental constituents o and \uparrow .

Curiously, Janik and Lukowski [30] computed the leading finite size correction to the Hubbard magnon energy at large $g = -t/U$, using instead the magnon-magnon scattering matrix and still got a sensible result (correct up to a factor of 2 — see equation (71) in [30]). Physically this makes sense because at strong coupling g — which from the Hubbard point of view corresponds to weak interaction strength U compared to the electron hopping kinetic energy t — the magnons are a weakly bound pair of o and \uparrow and when we scatter the magnon against another magnon we are effectively scattering it against two fundamental particles! However, as the coupling decreases, the magnon-magnon Luscher formula with BDS magnon-magnon S-matrix will no longer provide the correct Hubbard result.¹ As explained in section 2.2 the two results will agree to leading order — when the correction is of order $1/g$ — and start disagreeing at next to subleading order — at order $1/g^3$. Could we be in a similar situation in $\mathcal{N} = 4$ SYM? There, the finite size corrections [36–38] to the Giant Magnon [39] were reproduced at leading order [30, 31] and at next to leading order [32, 33] but no two loop computation is available. Bearing in mind what happens in the Hubbard model it is not completely inconceivable that at this loop order the Luscher results based on the lightcone S-matrix [11, 13] start failing. On the other hand from the string worldsheet point of view this scenario would certainly be intriguing.

In the opposite regime, at weak coupling g , expression (1.4) gives precisely the correct result whereas the use of the magnon-magnon BDS S-matrix is completely inappropriate because in this regime the elementary particles that make the magnon are highly bound. If this qualitative structure persists in the full $\mathcal{N} = 4$ theory then it explains why a naive computation of the Luscher terms at weak coupling seems to never yield any sort of transcendental numbers such as $\zeta(3)$ or $\zeta(5)$ which typically appear in the computations of [18, 19]. The reason would be that in order to probe the weak coupling limit of the theory the knowledge of the magnon constituents would be essential.²

¹Let us also remark that considering also the contribution from the S-matrix between magnon and bound state of magnons (which exist in the BDS theory) in the Luscher formalism does not cure this problem. At most, these contributions can reproduce the higher winding number diagrams of the fundamental constituents o and \uparrow only in the strong coupling regime $g \gg 1$. The reason is again that in this regime a bound state of b magnons is almost like $2b$ free fundamental particles.

²On the other hand, if the magnons in the light-cone gauged string theory can be thought of as fundamental particles — contrary to what happens in the Hubbard model — then a priori we should indeed

In particular it seems clear that a thermodynamic Bethe ansatz (TBA) approach to the BDS equations would *not* recover the Hubbard finite size corrections (because in particular the TBA approach always reduces to the Luscher formulae at large radius and as we explained those only work when we take into account the scattering of the magnons with the fundamental Hubbard electrons). In the context of the full $\mathcal{N} = 4$, the TBA program is being carried out in [40, 41] still with inconclusive results.

Still in the context of the Hubbard model we analyzed in sections 2.3.1 and 2.3.2 how wrapping interactions manifest themselves for many particle states. In the $\mathcal{N} = 4$ context this is an unavoidable question if we want to understand the full anomalous dimensions of small operators such as the Konishi operator (1.3). There are two different kind of effects one needs to take into account to study wrapping corrections to many particle states. On the one hand the energy of the state as a function of the magnon momenta changes when the theory is put in finite volume and this leads to a Luscher type correction which for the Hubbard magnons reads

$$\delta E_{\text{Luscher}} = \frac{1}{2} \sum_{n=1}^M \sum_{\sigma=o,\downarrow} \int_{\mathcal{C}_n} \frac{dq}{2\pi i} (\varepsilon'(q) - \varepsilon'_\infty(p_n)) e^{i(q-\phi_o)L} \prod_{m=1}^M S_{\phi,\sigma}^{\phi,\sigma}(p_m, q) + c.c.,$$

generalizing (1.4). Led by the striking simplicity of this expression we conjecture a generalization of the Luscher formula for many particle states in integrable two dimensional models in (2.38).

The second effect we need to take into account is the fact that due to the wrapping interactions the quantization conditions — that is the Bethe ansatz equations — for the magnon momenta receive corrections and thus the momenta are slightly shifted when wrapping interactions are taken into account. For example, BDS equations [4] can be *dressed* to

$$\left(\frac{x_n^+}{x_n^-} \right)^L = \prod_{m \neq n} \frac{u_n - u_m + i}{u_n - u_m - i} e^{i\phi_{nm}}$$

in such a way that the leading wrapping interactions are taken into account. Here ϕ_{nm} is a *wrapping dressing kernel* described in section 2.3.1.

As explained in section 2.3.3, a particularly curious feature of the computations of the weak coupling finite size corrections is that the leading wrapping correction, of order g^{2L} , to a state whose magnons' momenta are $p_j \simeq p_j^{(0)} + g^2 p_j^{(1)} + \dots + g^{2L} p_j^{(L)}$ only depends on the leading values $p_j^{(0)}$. This is natural from the point of view of the Luscher computations which are basically a smart way to organize the two dimensional perturbative expansion using the two dimensional S -matrix. In this formalism the exponentials which appear in the several integrands are automatically of order g^{2L} and thus the rest of the integrand, including the S -matrix and the dispersion relations, can be treated at g^0 order. In particular

sum over all these infinite number of bound states and of course infinite sums can eventually lead to transcendental numbers. This seems the only way out to find the good transcendental Konishi anomalous dimension from a weak coupling computation based on the Luscher formulae. We acknowledge N. Gromov, V. Kazakov and K. Zarembo for discussions on these issues.

we can easily compute the wrapping correction to any many magnon BDS state without performing any iteration of the BDS equations up to order g^{2L} . In $\mathcal{N} = 4$, this should be related to the fact that to compute the wrapping corrections to the Konishi operator we isolate the appropriate wrapping diagrams and only keep some information about the lower order graphs [17–19]. If the four dimensional wrapping Feynman diagrams in [17–20] could be re-written in a two dimensional language this could be the key to understanding the structure of an hypothetical hidden level of fundamental particles.

In section 3 we consider a completely different type of (toy) model where wrapping interactions are under control. Namely we study long ranged Hamiltonians coming from a transfer matrix algebraic construction *à la Leningrad school*. In the algebraic Bethe ansatz formalism the fundamental object is the transfer matrix, which is a trace of product of R -matrices,

$$T(u) = \text{tr}_{\text{aux}} R_L(u) \otimes \cdots \otimes R_1(u), \tag{1.5}$$

where the R -matrices are (simple) matrices obeying the Yang-Baxter relation and acting on the product of an auxiliary space (common for all R -matrices in this expression) and a physical space h_n . The full Hilbert space of a L -site spin chain is given by the tensor product

$$\mathcal{H} = h_1 \otimes \cdots \otimes h_L. \tag{1.6}$$

The transfer matrix is then an operator acting on the full Hilbert space (and by definition scalar w.r.t. the auxiliary space). The algebraic Bethe ansatz program yields us the spectrum of the transfer matrix $T(u)$. The diagonalization of this object is of great importance because, as we review in section 3, by taking n derivatives of the logarithm of the transfer matrix T at $u = 0$ we can generate integrable Hamiltonians of range n . In particular if we take more than L derivatives of this object we will start generating Hamiltonians which are long ranged, contain wrapping interactions and still, *by construction*, are integrable and diagonalized through a set of Bethe equations. For example if we consider

$$\hat{H} = \frac{1}{4i} \sum_{n=1}^{\infty} a_n \frac{g^{2n}}{n!} \frac{d^n}{d\lambda^n} \log \hat{T}(\lambda) \Big|_{\lambda=0} + h.c. \tag{1.7}$$

then we will have an Hamiltonian which at order g^{2n} is of range n and whose spectrum is given by a sum of individual dispersion relations plus a wrapping correction which starts precisely at order g^{2L} ,

$$\hat{H} = \sum_{j=1}^M \epsilon(p_j) + E_{\text{wrapping}}(p_1, \dots, p_M)$$

This behavior is probably completely generic and we considered Hamiltonians of the form (1.7) with $SU(2)$, $SU(N)$ and $SL(2)$ symmetry.

In the $SL(2)$ case we found the following curious behavior: suppose we consider a Hamiltonian of type (1.7) with a_n some algebraic numbers. Then the dispersion relation truncated at a given order g^{2n} is a rational function of these algebraic numbers and of the Bethe roots $u_j = \frac{1}{2} \cot \frac{p_j}{2}$, which are quantized via a set of polynomial equations. Thus

L	$u_{1,2}$	$\epsilon(p_1) + \epsilon(p_2)$	$E_{\text{wrapping}}(p_1, p_2)$
3	$\frac{\sqrt{3}}{5} \pm \frac{\sqrt{7}}{10}$	$\frac{11g^2}{8} + \frac{13\sqrt{3}g^4}{32} - \frac{5g^6}{6}$	$\frac{1}{32} (-26 + 3\pi^2 - 4\zeta(3)) g^6$
4	$\frac{1}{3} \pm \frac{\sqrt{7}}{6}$	$\frac{5g^2}{4} - \frac{g^4}{8} - \frac{19g^6}{24} + \frac{3g^8}{2}$	$\left(\frac{41}{32} - \frac{5\pi^2}{48} - \frac{\pi^4}{360}\right) g^8$
6	$\pm\frac{1}{2} + \frac{\sqrt{2}}{2}$	$g^2 + \frac{g^4}{\sqrt{2}} - \frac{g^6}{3} - \sqrt{2}g^8 - \frac{4g^{10}}{5} + \frac{5\sqrt{2}g^{12}}{3}$	$\frac{-1}{2\sqrt{2}} (7 - 4\zeta(3) - 2\zeta(5)) g^{12}$

Table 1: Energies of some two magnon states of a spin chain of length L and long range $SL(2)$ Hamiltonian given by equation (1.7) with $a_n = 1$. The total energy has contributions from the infinite volume dispersion relation $\epsilon(p)$ and from wrapping effects.

the contribution to the spectrum of the $\sum_{j=1}^M \epsilon(p_j)$ term will be given by some algebraic quantity. On the other hand, precisely at order g^{2L} the wrapping corrections enter the game and those are given by an infinite sum (3.17) of algebraic functions of the Bethe roots u_j . Typically they will give rise to transcendental contributions!

As an example, in table 1 we listed a couple of energies of some two magnon states up to order g^{2L} and for the simplest choice of $SL(2)$ Hamiltonian with $a_n = 1$. It would be very interesting to explore this connection between transfer matrices of non-compact groups and transcendentalities in the context of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Perhaps this could provide us with important hints about the origin of the dressing factor which is populated by transcendental numbers.

This paper is organized as follows: After introducing the Hubbard model in section 2 and reviewing the magnon description in section 2.1 we re-derive the exact finite volume dispersion relation by means of Feynman diagrams in section 2.1.1. In section 2.2 we explain how the leading finite size corrections can be understood from a Luscher type approach. In section 2.3 we study wrapping effects for general many particle states. In particular we review how the BDS equations follow from the Lieb-Wu equations, explain how they can be upgraded to include the leading wrapping corrections (section 2.3.1) and analyze the analogue of the Luscher formulas for many magnon states (section 2.3.2). Section 3 is devoted to the study of integrable long ranged Hamiltonians derived from an algebraic Bethe ansatz formalism and in section 3.1 we explore some generalizations of this construction and present a curious non-compact long ranged Hamiltonian where transcendentalities and wrapping are intimately related.

2. The Hubbard model

The one dimensional Hubbard model describes spin 1/2 electrons moving in a periodic lattice with L sites. The electrons can hop between neighboring sites and there is a repulsive (or attractive depending on the sign) potential when two electrons (with opposite spin) occupy the same lattice site. Obviously, due to Pauli exclusion principle no two equal spin electrons can ever occupy the same position. At half filling, when the number of electrons equals the number of sites, each electron will tend to occupy a site in the lattice due to the repulsive potential. We can then study an effective Hamiltonian for the spins alone [42]. It will be a long ranged Hamiltonian where the interactions correspond to *virtual*

processes where electrons hop there and back eventually changing spin in the process. In [23] this effective Hamiltonian was identified with the long range Hamiltonian of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This identification is correct up to three loops but fails beyond that. Still, this is an instructive toy model since wrapping interactions, due to electrons making loops around the ring, are perfectly under control. In this section we will study them, give them a diagrammatic description and understand how they fit into the usual field theoretical Luscher treatment of finite size corrections. We will also understand how to modify the effective Bethe equations for the magnons of the effective spin theory so that they reproduce the (leading) wrapping effects.

A quite useful alternative description of the relevant degrees of freedom in the Hubbard model is obtained by performing a so called Shiba duality. It amounts to thinking of the Hilbert space as that where $N - M$ vacancies o and M double occupancies \uparrow move in a ferromagnetic vacuum with L up spins.³ In this description the Hamiltonian reads

$$H = -t \sum_{i=1}^L \sum_{\sigma=o,\uparrow} \left(e^{i\phi_\sigma} a_{i,\sigma}^\dagger a_{i+1,\sigma} + h.c. \right) - U \sum_{i=1}^L a_{i,o}^\dagger a_{i,o} a_{i,\uparrow}^\dagger a_{i,\uparrow} \quad (2.1)$$

where, following [23], we have introduced some extra twists ϕ_σ in the Hamiltonian which can be thought of as a sort of magnetic flux inducing additional phases in the electron wave function as it moves around the chain. As explained in [22, 23] and reviewed below these twists can be used to delay the wrapping corrections to the effective spin theory.

The complete spectrum of this Hamiltonian can be obtained as

$$E = \sum_{n=1}^N \varepsilon(q_n) \quad , \quad \varepsilon(q) \equiv -2t \cos(q) \quad (2.2)$$

where the momenta are quantized through the solution of the Lieb-Wu [43] equations,

$$e^{i(q_n - \phi_o)L} = \prod_{j=1}^M \frac{u_j - 2g \sin(q_n) - i/2}{u_j - 2g \sin(q_n) + i/2} \quad , \quad n = 1, \dots, N \quad (2.3)$$

$$e^{i(\phi_o - \phi_\uparrow)L} \prod_{n=1}^N \frac{u_j - 2g \sin(q_n) - i/2}{u_j - 2g \sin(q_n) + i/2} = \prod_{k \neq j}^M \frac{u_j - u_k - i}{u_j - u_k + i} \quad , \quad j = 1, \dots, M \quad (2.4)$$

where

$$g = -\frac{t}{U}.$$

In section 2.3 we will review [23] how, eliminating the electron momenta q_n from these (twisted) Lieb-Wu equations, we obtain an effective set of (twisted) BAE for the spin degrees of freedom u which are precisely the (twisted) BDS equations [4]. In that section we will analyze wrapping corrections in full generality. We will see for example,

³The number of vacancies o and double occupancies \uparrow is related to the number of up and down spins as $N_\uparrow = N_\downarrow$, $L - N_o = N_\uparrow$.

that perturbatively in g the effective (twisted) Bethe ansatz equations are valid up to order g^{2L} for the choice of twists [23]

$$\phi_o - \phi_{\uparrow} - \frac{\pi}{L} = 0 \pmod{\frac{2\pi}{L}} \tag{2.5}$$

and the g^{2L} correction to any perturbative state will be given solely in terms of the position of the Bethe roots u_j to leading (g^0) order only. This is probably a simplifying feature of the weak coupling finite size corrections which is likely to be present in the $\mathcal{N} = 4$ supersymmetric spin chain.

We will also study the generic case where (2.5) does not hold since it will provide us with a nice toy model to understand Luscher type corrections for many particle states.

In the following two sections we will consider a much simpler setup which however captures most of the relevant physical information. Namely we will consider the very simple configuration with a single vacancy o and a single double occupancy $\uparrow\downarrow$, that is $N = 2$ and $M = 1$.

2.1 The magnon

A magnon in the Heisenberg XXX spin- $\frac{1}{2}$ chain,

$$H_{\text{xxx}} = \frac{1}{4} \sum_n (1 - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}) \tag{2.6}$$

is the lowest lying excitation above the ferromagnetic ground state. It is a plane wave state

$$\sum_n e^{ipn} \sigma_n^- | \uparrow \dots \uparrow \rangle$$

of one down spin in a chain of up spins, with excitation energy

$$\epsilon(p) = 1 - \cos p. \tag{2.7}$$

In the Hubbard model such plane wave is not an eigenstate of the Hamiltonian but there is a close analogue of this state when the empty site excitation o and the double occupancy $\uparrow\downarrow$ form a bound state (note that a o and a $\uparrow\downarrow$ in the same site is precisely the same as a spin down). More precisely, as reviewed in appendix A, we can consider the following half-filling state

$$|\Psi\rangle = \sum_{n,n'} \psi(n, n') \left(a_{n,\uparrow}^\dagger a_{n',o}^\dagger \right) | \uparrow \dots \uparrow \rangle.$$

with $\psi(n, n')$ being a superposition of plane waves with momenta q and q' . By acting with the Hamiltonian on this state we can see that there exist bound state solutions with⁴ $q, q' = \frac{p}{2} \pm i\beta$ where p is the bound state momentum while β dictates the exponential

⁴Throughout this paper we always use this definition of p which seems the most natural one from the Hubbard point of view. To make contact with the standard notations in the AdS/CFT literature we should use $p_{\text{here}} = p_{\text{usual}} + \pi$.

damping of the wave function away from $n = n'$. The energy of such states, which we also call magnons, equals

$$\epsilon(p) = -4t \cos \frac{p}{2} \cosh \beta. \quad (2.8)$$

In infinite volume we find $\beta = \beta_\infty(p)$ where

$$4g \cos \frac{p}{2} \sinh \beta_\infty(p) = -1 \quad (2.9)$$

so that $\epsilon(p) = \epsilon_\infty(p)$ with

$$\epsilon_\infty(p) = -\sqrt{U^2 + 16t^2 \cos^2 \frac{p}{2}} = -U (1 + 4g^2(\cos p + 1) + \dots) \quad (2.10)$$

which at weak coupling g has the same $-\cos p$ dependence as (2.7). This is expected from the known result that perturbatively in small $g = -t/U$ the Hubbard model at half filling is a long-ranged Hamiltonian whose leading term is the Heisenberg spin chain.

The magnon state can also be described by the triplet (q, q^*, u) satisfying the Lieb-Wu equations

$$\begin{aligned} e^{i(q-\phi_o)L} &= \frac{u - 2g \sin(q) - i/2}{u - 2g \sin(q) + i/2}, & e^{i(q^*-\phi_o)L} &= \frac{u - 2g \sin(q^*) - i/2}{u - 2g \sin(q^*) + i/2}, \\ e^{i(\phi_\uparrow-\phi_o)L} &\frac{u - 2g \sin(q) - i/2}{u - 2g \sin(q) + i/2} \frac{u - 2g \sin(q^*) - i/2}{u - 2g \sin(q^*) + i/2} &= 1, \end{aligned}$$

In infinite volume, the l.h.s of the first two equations is 0 and ∞ for a bound state with complex momentum $q = \frac{p}{2} + i\beta$. This fixes

$$u = 2g \sin q + \frac{i}{2}. \quad (2.11)$$

The reality of u implies

$$u = 2g \sin \frac{p}{2} \cosh \beta_\infty(p) \equiv u_\infty(p) \quad (2.12)$$

and the condition (2.9) which gives the dispersion relation (2.10).

When the state is put in finite volume, equation (2.8) is still valid but the expression (2.9) for $\beta(p)$ gets modified to

$$4g \cos \frac{p}{2} \sinh \beta = -\frac{\sinh \beta L}{\cosh \beta L - \cos \frac{L}{2} (p - 2\phi_o)} \quad (2.13)$$

In appendix A we derive this equation from the direct study of the magnon wave function in finite volume. Naturally, the same result can also be obtained from the Lieb-Wu equations (2.4). Indeed these Bethe equations are exact for any L since the interactions of the elementary particles are ultra local. The leading finite size correction to the magnon energy is then exponentially suppressed in the ratio of the system size L by the the physical size $1/\beta_\infty$ of the bound state,

$$\epsilon(p) - \epsilon_\infty(p) = U e^{-\beta_\infty(p)L} 2 \tanh \beta_\infty(p) \cos \frac{L}{2} (p - 2\phi_o) \quad (2.14)$$

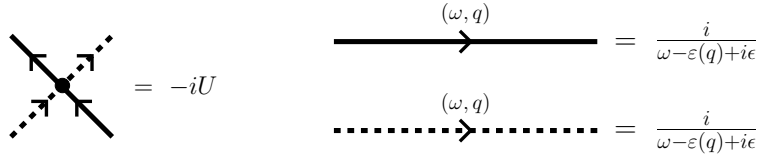


Figure 2: Feynman rules for diagrammatic computations in the Hubbard model. Each elementary particle (o and \uparrow) has a non-relativistic free propagator (solid and dashed lines) and there is only a quartic interaction vertex. Loops carry an extra minus sign as the elementary particles are fermions.

For the particular choice of twists (2.5), such that $e^{i(\phi_o - \phi_\uparrow)L} = -1$, this leading correction vanishes, and instead we have

$$\epsilon(p) - \epsilon_\infty(p) = U e^{-2\beta_\infty(p)L} 2 \tanh \beta_\infty(p) \quad (2.15)$$

In the next section we will recover these known results from a diagrammatic point of view. This will turn out to be very useful to understand how to recover the finite size corrections from the effective theory point of view, that is, from the BDS language. In particular we will understand that if we were given solely the BDS Bethe equations we would not be able to recover the finite size corrections using a Luscher type approach [34, 35] except at strong coupling. Instead, the knowledge of the magnon constituents when computing the virtual processes wrapping the space-time cylinder will turn out to be essential. The diagrammatic language seems therefore promising to try to learn some lessons about what to expect for the $\mathcal{N} = 4$ SYM chain if this chain most fundamental description comprises extra degrees of freedom [23–28].

2.1.1 Diagrammatically

In this section we shall explore the field theoretic description of the Hubbard model defined by the Hamiltonian (2.1). This will allow us to use the powerful diagrammatic techniques of field theory to obtain the finite size effects from loop diagrams with topological winding around the compact direction. We start by writing the action of the theory

$$S = \int dt \sum_n (\mathcal{L}_o + \mathcal{L}_\uparrow + \mathcal{L}_{\text{int}})$$

where the free part is given by

$$\mathcal{L}_\sigma = \frac{i}{2} (a_{n,\sigma}^* \partial_t a_{n,\sigma} - a_{n,\sigma} \partial_t a_{n,\sigma}^*) + t \left(e^{i\phi_\sigma} a_{n,\sigma}^* a_{n+1,\sigma} + e^{-i\phi_\sigma} a_{n,\sigma}^* a_{n-1,\sigma} \right)$$

and the interaction by

$$\mathcal{L}_{\text{int}} = U a_{n,o}^* a_{n,o} a_{n,\uparrow}^* a_{n,\uparrow}$$

The elementary excitations o and \uparrow have a non-relativistic propagator

$$\langle T a_{n,\sigma}(t) a_{n',\sigma}^*(t') \rangle = \int \frac{d\omega}{2\pi} \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-i\omega(t-t') + i(q - \phi_\sigma)(n-n')} \frac{i}{\omega - \epsilon(q) + i\epsilon}$$



Figure 3: Feynman diagrams for the two point function of the composite operator χ .

where

$$\varepsilon(q) = -2t \cos(q).$$

The interaction term in the action gives rise to a quartic coupling with coupling constant U . The Feynman rules are summarized in figure 2. In a non-relativistic theory the number of particles is conserved and this greatly simplifies the field theoretic perturbative expansion of the theory [44–46]. Diagrammatically, this fact stems from the retarded nature of the propagators which gives them an orientation.

In order to find the two particle spectrum we consider the two point function

$$\langle T \chi_n(t) \chi_{n'}(t') \rangle = \int \frac{d\Omega}{2\pi} \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{-i\Omega(t-t') + ip(n-n')} G(\Omega, p)$$

of the composite operator

$$\chi_n(t) = a_{n,o}(t) a_{n,\downarrow}(t) + a_{n,o}^*(t) a_{n,\uparrow}^*(t)$$

In particular, to find out possible bound states (magnons) we should look for poles of $G(\Omega, p)$ thus obtaining the dispersion relation $\Omega(p)$. Notice that there is a big arbitrariness in the choice of the composite operator χ . The only requirement is that the state it creates has some overlap with the magnon wave function.

The propagator $G(\Omega, p)$ can be computed diagrammatically. It is given by the sum of all diagrams describing the two elementary particles moving freely and interacting k times as shown in figure 3. More precisely, it is given by

$$G(\Omega, p) = \sum_{k=0}^{\infty} [G_0(\Omega, p)]^{k+1} (-iU)^k = \frac{i}{iG_0^{-1}(\Omega, p) - U} \quad (2.16)$$

where the free propagator

$$G_0(\Omega, p) = - \int \frac{d\omega}{2\pi} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{i}{\omega - \varepsilon(q) + i\epsilon} \frac{i}{\Omega - \omega - \varepsilon(p-q) + i\epsilon}$$

can be computed by residues,

$$G_0(\Omega, p) = \frac{i}{\sqrt{\Omega^2 - 16t^2 \cos^2 \frac{p}{2}}}.$$

Thus, the full propagator reads

$$G(\Omega, p) = \frac{i}{\sqrt{\Omega^2 - 16t^2 \cos^2 \frac{p}{2}} - U}$$

and the magnon bound state corresponds to the pole at

$$\Omega = -U\sqrt{1 + 16g^2 \cos^2 \frac{p}{2}}$$

which is precisely the infinite volume result (2.10).

With periodic boundary conditions of size L the free propagator is changed. The natural way to account for this effect is to sum over all possible windings of the loops in figure 3 around the compact circle. The winding numbers are a topological property of the Feynman graph that can be assigned to any of the propagators forming each loop. We shall compute the graph assigning the winding number m always to the particle o . Furthermore, equation (2.16) remains valid provided

$$\begin{aligned} G_0(\Omega, p) &= - \sum_{m=-\infty}^{\infty} \int \frac{d\omega}{2\pi} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{i}{\omega - \varepsilon(q) + i\epsilon} \frac{i}{\Omega - \omega - \varepsilon(p - q) + i\epsilon} e^{im(q - \phi_o)L} \\ &= \frac{iF_L(\Omega, p)}{\sqrt{\Omega^2 - 16t^2 \cos^2 \frac{p}{2}}} \end{aligned}$$

where, if we parametrize Ω as

$$\Omega = -4t \cos \frac{p}{2} \cosh \beta, \tag{2.17}$$

we get

$$F_L(\Omega, p) = 1 + \sum_{m=1}^{\infty} e^{-mL\beta} \cos mL \left(\frac{p}{2} - \phi_o \right) = \frac{\sinh(\beta L)}{\cosh(\beta L) - \cos L \left(\frac{p}{2} - \phi_o \right)}$$

The full propagator then becomes

$$G(\Omega, p) = \frac{iF_L(\Omega, p)}{\sqrt{\Omega^2 - 16t^2 \cos^2 \frac{p}{2} - UF_L(\Omega, p)}}$$

and the magnon pole sits at (2.17) with β determined from the equation

$$4t \cos \frac{p}{2} \sinh \beta = UF_L(\Omega, p)$$

which is precisely (2.13).

2.2 Luscher in Hubbard

Luscher developed a general formalism [34, 35, 30] to determine the leading finite size corrections in quantum field theory. In particular, he studied the change in the dispersion relation of one particle when one imposes periodic boundary conditions. Remarkably, he found that the leading correction to the energy $\epsilon(p)$ of a particle with momentum p was completely fixed by the particles infinite volume dispersions relations and S-matrix. The idea is that the on-shell dispersion relation is defined by the pole of the propagator. Following the standard notation of relativistic field theory we can write

$$iG^{-1}(\Omega, p) = -\Omega^2 + \epsilon_{\infty}^2(p) + \Sigma(\Omega, p) = 0 \tag{2.18}$$

where

$$\Sigma(\Omega, p) = \Sigma_\infty(\Omega, p) + \delta\Sigma(\Omega, p)$$

is the particle's self-energy, whose infinite volume part vanishes with zero derivative on the mass-shell $\Omega = \epsilon_\infty(p)$. From (2.18) with $\Omega = \epsilon_\infty(p) + \delta\epsilon(p)$ we obtain

$$\delta\epsilon(p) = \frac{\delta\Sigma}{2\epsilon_\infty} + \left(\frac{\delta\Sigma}{2\epsilon_\infty}\right) \left(\frac{\partial_0\delta\Sigma}{2\epsilon_\infty}\right) + \left(\frac{1}{2}\partial_0^2\Sigma - 1\right) \frac{1}{2\epsilon_\infty} \left(\frac{\delta\Sigma}{2\epsilon_\infty}\right)^2 + \dots \quad (2.19)$$

with all the quantities computed at the infinite volume mass-shell $\Omega = \epsilon_\infty(p)$. The self-energy correction $\delta\Sigma$ can then be related to the S-matrix. This is achieved by evaluating the self-energy diagrams with winding around the compact direction by deforming the integration over the loop momenta to pick the on-shell pole of the wound internal propagator and obtain an integral of the forward scattering S-matrix [34, 35, 30]. More precisely, the leading finite size correction to $\epsilon^a(p)$ is given by⁵

$$\frac{1}{2} \int_{\mathcal{C}} \frac{dq}{2\pi i} e^{iLq} \sum_b (-1)^{F_b} \left(\epsilon_\infty^{b'}(q) - \epsilon_\infty^{a'}(p) \right) \left(1 - S_{ab}^{a'b}(p, q) \right) + c.c.$$

where $S_{ab}^{a'b}(p, q)$ is the S-matrix for forward scattering of particle a with particle b , $F_b = \pm 1$ encodes the bosonic/fermionic nature of particle b and we consider only the contribution from diagrams with winding number ± 1 . For usual relativistic theories the contour \mathcal{C} is given by an integral over the possible S -matrix poles plus an integral over the real axis. The latter describes a quantum loop and is absent in our case where the underlying theory is non-relativistic [44–46]. The former can be simply computed by residues and is denoted by μ -term.

Let us now apply this general formalism to the Hubbard model and find the leading finite size correction to the magnon dispersion relation. As we saw in the previous sections the magnon is a bound state with a finite size $1/\beta_\infty$. Therefore, its energy in a finite system has a leading correction of order $e^{-\beta_\infty L}$, except for a particular choice of twists where this correction can be delayed to order $e^{-2\beta_\infty L}$. The diagram leading to the first wrapping correction corresponds to the splitting of the magnon into its fundamental constituents each one going around the space-time cylinder in opposite directions and meeting on the other side of the cylinder as depicted in figure 4. The Luscher μ -term then reads

$$\delta\epsilon(p) = \sum_{\sigma=o,\downarrow} \left(\frac{1}{2} \text{Res}_q e^{iL(q-\phi_\sigma)} (\epsilon'(q) - \epsilon'_\infty(p)) S_{\phi,\sigma}^{\phi,\sigma}(p, q) + c.c. \right) \quad (2.20)$$

⁵In [34, 35] this formula was derived for relativistic theories and $p = 0$. This amounts to computing the corrections to the particle's mass. In the process of derivation the propagator is wound around the spacial circle originating a factor of $\cos(qL)$ multiplying the infinite volume propagator. Since the spacial momentum p vanished the problem was isotropic with respect to the spacial directions and this factor could be simply replaced by $2e^{iqL}$. In the case $p \neq 0$ we *must* treat each exponential separately which amounts to summing the contributions from the virtual particles going parallel and anti-parallel to the physical particle. In most cases, including those considered in this paper, these symmetrizations will simply lead to computing the real part of the result obtained keeping one exponential.

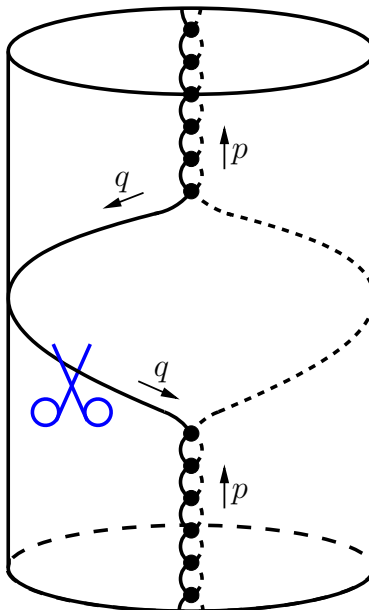


Figure 4: The leading finite size correction to the magnon energy is given by the Feynman diagram where its two elementary constituents split and merge after winding the spacetime cylinder once. After cutting the wound loop, putting one elementary particle on shell, this diagram gives rise to the Luscher μ -term.

where q and $\varepsilon(q)$ are the momentum and energy of the elementary particle o going around the loop in figure 4, $\epsilon_\infty(p)$ is the infinite volume dispersion relation of the magnon and the residue is taken at the pole of the S -matrix. The S -matrix between magnon and elementary particle (o or \uparrow) can be read of from the Bethe equations (2.4)

$$S_{\uparrow, o}^{\uparrow, o}(p, q) = S_{\uparrow, \uparrow}^{\uparrow, \uparrow}(p, q) = \frac{u_\infty(p) - 2g \sin(q) - i/2}{u_\infty(p) - 2g \sin(q) + i/2}$$

with

$$u_\infty(p) = 2g \sin \frac{p}{2} \cosh \beta_\infty(p).$$

The pole condition

$$2g \sin(q) = u_\infty(p) + i/2$$

has the simple solution

$$q = \frac{p}{2} + i\beta_\infty(p).$$

Putting everything together we recover the result (2.14). For the choice of twists (2.5) this leading term vanishes and one needs to consider the contribution coming from diagrams

with winding number ± 2 ,⁶

$$\delta\epsilon(p) = \text{Res}_q e^{i2L(q-\phi_o)} (\epsilon'(q) - \epsilon'_\infty(p)) S_{\phi,o}^{\hat{\phi},o}(p, q) + c.c.$$

which gives (2.15).

2.2.1 Luscher with magnon-magnon S-matrix

In the previous sections we saw, by many means including a Luscher type computation, that the leading finite size correction to the magnon dispersion relation is given by [23]

$$\delta\epsilon(p) = U e^{-2\beta_\infty L} 2 \tanh \beta_\infty$$

when the Hubbard twists are chosen as in (2.5). This expression is valid for all values of the coupling g . In particular, at strong coupling we find [37]

$$\delta\epsilon(p) = \left(-\frac{U}{2g} \sec \frac{p}{2} + \frac{U}{64g^3} \sec^3 \frac{p}{2} + \dots \right) \exp -\frac{L}{2g} \sec \frac{p}{2} \left(1 - \frac{1}{96g} \sec^2 \frac{p}{2} + \dots \right). \quad (2.21)$$

In [30] — in the process of computing the leading μ -term prefactor for the $AdS_5 \times S^5$ giant magnon — Janik and Lukowski also determined what the contribution would be for the BDS scenario. They found

$$\delta\epsilon(p) = -\frac{U}{g} \sec \frac{p}{2} e^{-\frac{L}{2g} \sec \frac{p}{2}} + \dots$$

where we adapted their results to our normalizations. This result was obtained applying Luscher formula with the magnon-magnon S-matrix and thus describes a different physical process from the point of view of the elementary particles. Surprisingly, the two results agree up to an overall factor of 2. Physically one can try to justify this result using the fact that for large g the magnon is a weakly bound state of the elementary particles o and \uparrow . Thus, it is not unreasonable that the effect of a magnon loop around the spacetime cylinder is almost the same as a loop of the elementary particles with winding number 2. In any case, the accidental nature of this agreement is confirmed by its limitation to the strong coupling regime. In appendix C we repeat the computation of [30], expanding the result further in $1/g$,

$$\delta\epsilon(p) = \Re \left(-\frac{U}{g} \sec \frac{p}{2} + \frac{iU}{g^2} \tan \frac{p}{2} \sec^2 \frac{p}{2} + \dots \right) \exp -\frac{L}{2g} \sec \frac{p}{2} \left(1 - \frac{i}{2g} \tan \frac{p}{2} \sec \frac{p}{2} + \dots \right). \quad (2.22)$$

So that the result starts differing (apart from the factor of 2) at the next order in $1/g$. If we fix L/g and expand both the prefactor and exponent then we can easily see that the $1/g^2$ term drops out and the mismatch is delayed to the next to leading order in $1/g$, as

⁶For generic twists, the subleading correction (of order $e^{-2\beta_\infty L}$) to the magnon dispersion relation depends on all terms in (2.19), including the second derivative of the infinite volume self-energy which is not a physical on-shell observable. However if $\delta\Sigma$ vanishes at order $e^{-\beta_\infty L}$ then we only need to compute the first term, that is, the correction to the self energy due to winding 2.

mentioned in the introduction. This limit is the one usually considered in the AdS/CFT context.

The BDS magnon effective theory, reviewed in the next section, contains bound states of magnons [47] which are simple generalizations of the Bethe strings in the Heisenberg model. In [31] the Luscher term accounting for the finite size correction to the (dyonic) Giant magnon [47] dispersion relation was written including the virtual exchange of all these bound states of magnons. Thus, one might question if the sum of the contributions of all these bound states will correct the result (2.22) to give the exact result (2.21). This is *not* the case. In fact all diagrams present in the theory are those in figure 3 and they do *not*, in any sense, describe virtual exchanges of magnons and magnon bound states. At most, at very large g the loops of b magnons could be mimicking $2b$ loops of fundamental particles and thus, the best we can expect from summing all the possible magnons is to recover the *full finite volume result* but always for $g \rightarrow \infty$.

Furthermore, in the weak coupling regime the magnon-magnon computation gives the right g^{2L} coupling dependence but misses completely the momentum dependence of $\delta\epsilon$.

This result puts in question the validity of direct application of Luscher formulas to the giant magnon of $\mathcal{N} = 4$ SYM. In particular, if the giant magnon is not an elementary particle (like the 2 electrons bound state in the Hubbard model) of the worldsheet theory then its finite volume energy will be sensitive to the elementary excitations of the theory and can not be recovered just from the magnon-magnon S-matrix in infinite volume. On the positive side, this makes the wrapping effects a window into the most elementary level of the theory.

2.3 Magnon effective Bethe ansatz equations and generic wrappings

In this section we will review the results of [23] and see how the Bethe equations for the effective system of spins at half-filling coincides with the (twisted) BDS equations. We will then study the wrapping interactions for a general state. In particular we will understand how to *dress* the BDS equations in such a way that they include the wrapping interactions. We will also see that the leading order correction to the energy of a generic M magnon state can be given a Luscher type diagrammatic interpretation.

At half filling the Bethe ansatz equations (2.4) are given by

$$e^{i(q_n - \phi_o)L} = \prod_{j=1}^M \frac{u_j - 2g \sin(q_n) - i/2}{u_j - 2g \sin(q_n) + i/2}, \quad (2.23)$$

$$e^{i(q_{n+M} - \phi_o)L} = \prod_{j=1}^M \frac{u_j - 2g \sin(q_{n+M}) - i/2}{u_j - 2g \sin(q_{n+M}) + i/2}, \quad (2.24)$$

$$e^{i(\phi_{\uparrow} - \phi_o)L} \prod_{n=1}^M \frac{u_j - 2g \sin(q_n) - i/2}{u_j - 2g \sin(q_n) + i/2} \frac{u_j - 2g \sin(q_{M+n}) - i/2}{u_j - 2g \sin(q_{M+n}) + i/2} = \prod_{k \neq j}^M \frac{u_j - u_k - i}{u_j - u_k + i}, \quad (2.25)$$

where we explicitly split the $2M$ momenta q_n into two equal groups so that all indices range from 1 to M . A state with M magnons is a state where the $3M$ Bethe roots organize

in triplets of Takahashi states

$$(q_n, q_{n+M}, u_n), \tag{2.26}$$

with

$$q_{n+M} = q_n^*,$$

while the real u root is given by

$$u_n \simeq 2g \sin q_n + \frac{i}{2} = 2g \sin q_n^* - \frac{i}{2}. \tag{2.27}$$

The last condition is necessary once we allow q_n to have a positive imaginary part (and thus $q_{n+M} = q_n^*$ to have a negative imaginary part). In this case, the l.h.s of equations (2.23) and (2.24), respectively, vanishes and diverges exponentially with the system size L and condition (2.27) is required to obtain the same behavior for the r.h.s. It is also clear - and shown in [23] - that if we multiply the equations for each element of the Takahashi triplet we obtain the twisted BDS equations

$$e^{i(p_n - \phi_o - \phi_\dagger)L} \simeq \prod_{m \neq n}^M \frac{u_n - u_m + i}{u_n - u_m - i}, \tag{2.28}$$

parametrizing the momenta as

$$q_n = \frac{p_n}{2} + i\beta_n.$$

The dispersion relation $\epsilon(q_n) + \epsilon(q_{n+M})$ as function of p and the relation between p and the Bethe roots u is to leading order exactly as before, as it follows simply from computing the real and imaginary part of (2.27), namely $\beta_n \simeq \beta_\infty(p_n)$ given in (2.9),

$$u_n \simeq u_\infty(p_n) \equiv \frac{1}{2} \tan \frac{p_n}{2} \sqrt{1 + 16g^2 \cos^2 \frac{p_n}{2}}, \tag{2.29}$$

and $E \simeq \sum \epsilon_\infty(p_n)$ with

$$\epsilon_\infty(p) \equiv -\sqrt{U^2 + 16t^2 \cos^2 \frac{p}{2}}. \tag{2.30}$$

2.3.1 Corrected BAE

In this section we will study expressions (2.28), (2.29), (2.30) in greater detail. That is we will understand how these relations get modified once the leading finite size effects are taken into account. To do so we will study the leading wrapping effects for a generic many magnon state.

Physically there are two sources of corrections to the energy of a many magnon state. On the one hand, the energy of the state as a function of the magnon momenta changes when the state is put in finite volume. This will lead to a Luscher type correction. On the other hand, the periodicity condition, that is the BAE, are corrected and thus the quantized momenta are shifted. Diagrammatically this last effect would be due to new wrapping virtual processes correcting the magnon S-matrix rather than to the usual self energy virtual graphs present for the Luscher type contribution.

As for the single magnon case we will see that the bound state structure of the magnon must be taken into account to reproduce the proper finite volume results.

Needless to say, instead of correcting the effective BDS equations we could simply use the exact Lieb-Wu nested Bethe equations! The point is that we want to understand how the corrections to the Bethe equations of effective spin theories look like. This might be useful if, as discussed in the introduction, the $\mathcal{N} = 4$ SYM long-ranged Hamiltonian stems from an underlying Hubbard like description.

We want to eliminate the magnon momenta q_n from the Lieb-Wu equations thus obtaining an effective equation for the magnon rapidities u_n . As we saw, to leading order we simply have (2.27) but since we want also to keep track of the leading wrapping corrections to the effective equations we should instead write $q_n = \frac{p_n}{2} + i\beta_n$ and

$$u_n = 2g \sin q_n + \frac{i}{2} + \Delta_n = 2g \sin q_n^* - \frac{i}{2} + \Delta_n^* \quad (2.31)$$

where Δ_n is a small quantity which can be computed from the equations (2.23) and (2.24) for the magnon momenta. This is done in appendix B. Since u_n is real we can compute both u_n and β_n from the knowledge of the real and imaginary of the small quantity Δ_n . In particular, taking the imaginary part of (2.31), we obtain that $\delta\beta_n \equiv \beta_n - \beta_\infty(p_n)$ is given by

$$\delta\beta_n \simeq 2 \tanh \beta_\infty(p_n) \Im(\Delta_n) \quad (2.32)$$

where as before $\beta_\infty(p)$ is defined through (2.9). To proceed we introduce the notation $q_n^\infty \equiv \frac{p_n}{2} + i\beta_\infty(p_n)$ and $u_n^\infty \equiv 2g \sin q_n^\infty - \frac{i}{2}$ so that

$$u_n = u_n^\infty + 2g i \cos q_n^\infty \delta\beta_n + \Delta_n + \mathcal{O}(e^{-2\beta L}), \quad (2.33)$$

Next, as explained in greater detail in appendix B, we multiply the three Lieb-Wu Bethe equations for the Takahashi triplet (2.26) and expand using (2.31) and (2.33) to find

$$e^{i(p_n - \phi_\uparrow - \phi_\circ)L} = \prod_{m \neq n} \frac{u_n^\infty - u_m^\infty + i}{u_n^\infty - u_m^\infty - i} e^{i\phi_{nm}} + \mathcal{O}(e^{-2\beta L}), \quad (2.34)$$

where the *wrapping dressing kernel* ϕ_{nm} is given by⁷

$$\phi_{nm} = -\frac{\Im(\Delta_n)}{(u_n^\infty - u_m^\infty)^2 + 1} \left(\frac{1}{u_n^\infty} \tan^2 \frac{p_n}{2} + \frac{2}{u_n^\infty - u_m^\infty} \right) - (n \leftrightarrow m) \quad (2.35)$$

The dressed Bethe equations (2.34) can trivially be solved perturbatively provided a solution p_n^∞ to the original BDS equations is given. In this case u_n^∞ , which was a *function* of p_n , can be expanded around the value $p_n = p_n^\infty$. The value of the Bethe roots for these values of momenta are denoted by \mathbf{u}_n^∞ . We stress again, the \mathbf{u}_n^∞ are the values of the Bethe roots obtained via the BDS equations whereas u_n^∞ are *functions* of a free variable

⁷If the twists are chosen as in (2.5) the imaginary part of Δ_n becomes of order $e^{-2\beta L}$ and we need to expand further. In this case we obtain (B.6) in appendix B and (2.33), (2.34) hold to order $e^{-3\beta L}$.

p_n . Writing $u_n^\infty = \mathbf{u}_n^\infty + \delta u_n$ we easily see that the leading order shifts to the Bethe roots due to the inclusion of the dressing Kernel reduces to the simple linear problem

$$L\delta p_n - \sum_m \frac{2(\delta u_m - \delta u_n)}{(\mathbf{u}_n^\infty - \mathbf{u}_m^\infty)^2 + 1} = \sum_{m \neq n} \phi_{n,m}$$

with $\delta u_n = \left(\frac{d u_n^\infty}{d p_n}\right)_{p_n=p_n^\infty} \delta p_n$. Having found δp_n and $\delta \beta_n$ we want to compute the shift to the many particle state energy

$$\delta E = \sum_n -4t \cos \frac{p_n}{2} \cosh \beta_n + 4t \sin \frac{p_n^\infty}{2} \cosh \beta_\infty(p_n^\infty)$$

due to the wrapping effects. As mentioned above, this expression is non-zero due to two completely distinct type of corrections.

On the one hand, as we just saw, the momenta are quantized differently and thus we will have a contribution due to the displacement of the Bethe roots when wrapping interactions are taken into account. For many particles states like the ones we are now considering these corrections must be taken into account.

On the other hand the functional dependence of the energy on the momenta $\{p_n\}$ changes when we put the system in finite volume. More precisely β_n will be given by the infinite value expression $\beta_\infty(p_n)$ plus the correction $\delta \beta_n$ which will in general depend on all the magnon momenta in an entangled way. This contribution is precisely the analogue of the Luscher corrections described in the previous sections for the single magnon case. We will discuss these corrections in greater detail in the next subsection.

Putting these two corrections together we immediately get

$$\delta E = \sum_n \left(-U \delta \beta_n + \frac{d \epsilon_\infty(p_n)}{d p_n} \delta p_n \right). \tag{2.36}$$

Roughly speaking we could say that the first term is of Luscher type and accounts for virtual processes correcting the magnon dispersion relations while the second term stems from correcting the magnon S -matrix and therefore the BAE. It would be interesting to provide ϕ_{nm} with a diagrammatic interpretation. Moreover, in the thermodynamical Bethe ansatz approach to the computation of finite size effects in relativistic integrable models, *renormalized* Bethe equations for the positions of extra zeros in the TBA Y -system appear [48]. If the same structure emerges for the AdS/CFT TBA equations then the form of the above dressed equations (2.34) might provide some hints about the possible aspect of such dressed equations.

2.3.2 Meeting (and generalizing) Luscher

In this section we want to analyze the first correction $\delta E_{\text{Luscher}} = -U \sum_{n=1}^M \delta \beta_n$ to the energy of a M magnon state when put in finite volume and provide it with a simple physical diagrammatic meaning.⁸ Using (2.32) and the expression for Δ_n in appendix B we find

$$\delta E_{\text{Luscher}} = \frac{U}{2} \sum_{n=1}^M 2 \tanh \beta_\infty(p_n) e^{i(\frac{p_n}{2} + i\beta_\infty(p_n) - \phi_o)L} \prod_{m \neq n}^M \frac{\mathbf{u}_m^\infty - \mathbf{u}_n^\infty + i}{\mathbf{u}_m^\infty - \mathbf{u}_n^\infty} + c.c.$$

⁸This section, together with appendix D benefited largely from discussions with K. Zarembo.

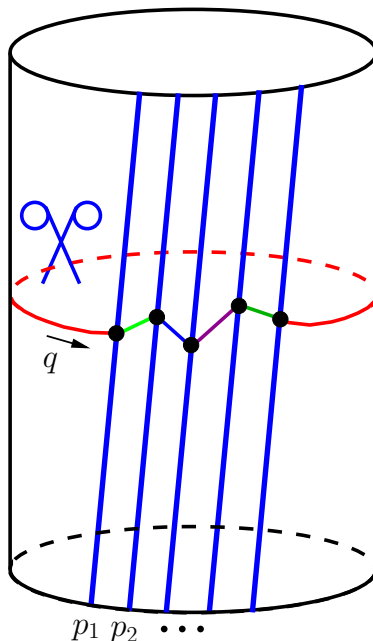


Figure 5: The many particle state is corrected due to interactions with a virtual particle going around the spacetime cylinder. For integrable theories the correction to the energy of the state is expressed in terms of a product of factorized scattering matrices between the virtual particle and the various physical particles. For non-diagonal scattering this product defines a transfer matrix, a central object in quantum integrability.

which can be written as

$$\delta E_{\text{Luscher}} = \frac{1}{2} \sum_{n=1}^M \sum_{\sigma=o,\uparrow} \int_{\mathcal{C}_n} \frac{dq}{2\pi i} (\varepsilon'(q) - \varepsilon'_\infty(p_n)) e^{i(q-\phi_o)L} \prod_{m=1}^M S_{\phi,\sigma}^{\hat{\phi},\sigma}(p_m, q) + c.c. \quad (2.37)$$

where \mathcal{C}_n encircles the pole of $S_{\phi,\sigma}^{\hat{\phi},\sigma}(p_n, q)$ at $q = \frac{p_n}{2} + i\beta_\infty(p_n)$ in the counter-clockwise direction. Obviously this expression resembles the single magnon Luscher formula (2.20) used before and can be thought of as its many particle generalization. Physically it represents the correction to the state self-energy due to the process where a virtual particle with momenta q goes around the cylinder scattering with all other physical excitations as depicted in figure 5.

Due to its strikingly simple form one might try to guess what the many particle Luscher formula for generic quantum integrable two dimensional field theories with factorized scattering could be. A likely candidate for such expression for a state with M particles with polarizations a_1, \dots, a_M and momenta p_1, \dots, p_M would be

$$\delta E_{\text{Luscher}} = \Re \left\{ \sum_{n=1}^M \sum_{\{b_1, \dots, b_M\}} \int \frac{dq}{2\pi i} (\varepsilon'_{b_n}(q) - \varepsilon'_{a_n}(p_n)) e^{i(q-\phi_{b_1})L} (-1)^{F_{b_1}} \right. \quad (2.38)$$

$$\left. \left[S_{a_1, b_1}^{a_1, b_2}(p_1, q) S_{a_2, b_2}^{a_1, b_3}(p_2, q) \dots S_{a_M, b_M}^{a_M, b_1}(p_M, q) - \delta_{b_1}^{b_2} \dots \delta_{b_M}^{b_1} \right] \right\}$$

where we sum over the fundamental polarization b_j but in principle allow the physical particle to be bound states in which case some of the indices a_n will be bound states indices as in (2.37). In this case, the corresponding S-matrices in this expression should be the usual fused S -matrices. For fermionic virtual particle we include the standard -1 factor from the loop and in case magnetic fields are coupled to any of the particles a corresponding twist is included. Notice that we allow the polarization of the virtual particle to change as it scatters with each of the physical particles. This is not in contradiction with (2.37) because due to charge conservation $S_{\phi,\sigma}^{\phi,\sigma'} = 0$ if $\sigma \neq \sigma'$. We also notice that, not surprisingly, the first term in the second line can be written in a very compact form in terms of the transfer matrix $\hat{T}(q) = \text{str}_0 \left(\hat{S}_{1,0}(p_1, q) \dots \hat{S}_{M,0}(p_M, q) \right)$, a central object in integrable theories. Finally the second term in the last line is irrelevant if we only integrate over the S-matrices poles but for relativistic theories we expect the forward scattering amplitude to appear when the momenta is integrated over the real axis.

In the AdS/CFT setup it would be interesting to consider this expression applied to the computation of the exponential corrections to spinning strings [49–51]. In the scaling limit the transfer matrix becomes the exponential of the algebraic curve quasimomenta [5, 6, 52–54] and using the techniques in [27, 32, 55–58] one might try to study semi-classical wrapping corrections around fairly general classical solutions.

Finally, we should stress that expression (2.38) is a conjecture for which we have no prove but only empirical evidence. It would be interesting to try to directly generalize Luscher arguments for many particles states in integrable theories in which factorizability should provide dramatic simplifications.

2.3.3 Perturbative treatment

In this section we will consider the perturbative small g regime. Since

$$e^{-\beta_\infty} = 2g \cos \frac{p}{2} + \mathcal{O}(g^3)$$

the leading finite size corrections for generic twists will appear at order g^L and at order g^{2L} for twists given by (2.5). For example the single magnon results of section 2.1 are easily seen to give

$$\epsilon(p) - \epsilon_\infty(p) \simeq \begin{cases} -2U \left(2g \cos \frac{p}{2} \right)^L \cos \frac{L}{2}(p - \phi_o) , & \text{for generic twists} \\ -2U \left(2g \cos \frac{p}{2} \right)^{2L} , & \text{for twists as in (2.5)} \end{cases} \quad (2.39)$$

As explained in section 2.3.1, when we want to consider states with more than one magnon we expect two types of contributions. On the one hand, we have a Luscher type contribution of the form

$$\delta E_{\text{Luscher}} \simeq -U \sum_{n=1}^M \delta \beta_n \quad (2.40)$$

due to the fact that the energy of the state, as a function of the magnon momenta, changes when the state is put in finite volume. On the other hand we obtain extra corrections due

to the fact that the effective Bethe equation for the magnons are corrected when wrapping effects are taken into account,

$$\delta E_{BAE} \simeq \sum_{n=1}^M \frac{d\epsilon_\infty(p_n)}{dp_n} \delta p_n.$$

However, since perturbatively the derivative of the dispersion relation with respect to p carries an extra power of g^2 , (2.40) is sufficient to compute the leading wrapping correction. Notice that since the exponential factors of $e^{-\beta\infty L}$ start at g^L order the prefactors can be computed using the g^0 order positions of the Bethe roots u_n^∞ (which we simply denote by u_n in what follows). This is a huge simplification which is probably also present in $\mathcal{N} = 4$ SYM if the most fundamental description of the supersymmetric chain bears a resemblance with the Hubbard model.

Generic states can be studied by expanding at small g the expression in appendix B. For concreteness let us focus on 2 magnon states with twists given by (2.5). Moreover we chose the twists as given in [23] which correspond not only to (2.5) but also to $e^{i\phi_1 L} = (-1)^{L+1}$. In this case not only the wrapping order is delayed to g^{2L} but also the BDS effective equations become untwisted. We obtain, to leading g^{2L} order,

$$\frac{\delta E}{U} = 2 \left(\Delta_1^{(0)}\right)^2 + 2 \left(\Delta_2^{(0)}\right)^2 + \frac{4 \Delta_1^{(0)} \Delta_2^{(0)}}{(u_1 - u_2)^2 + 1} + o(g^{2L}) \tag{2.41}$$

with

$$\Delta_{1,2}^{(0)} = ig^L \left(\frac{i}{u_{1,2} - i/2} \right)^L \frac{u_{1,2} - u_{2,1} - i}{u_{1,2} - u_{2,1}}.$$

For Konishi like states with opposite momenta p and $-p$ (and thus $u_1 = -u_2$) we get the remarkably simple expression

$$\frac{\delta E}{U} = 2^{2L+1} g^{2L} (\cos(p) + 3) \csc^2 \frac{p}{2} \cos^{2L} \frac{p}{2}. \tag{2.42}$$

In particular, we might consider some "high energy magnons" with $p \simeq -\pi$ in a long spin chain to find

$$\delta E^{\text{large momentum}} \sim (p + \pi)^{2L} \ll 1$$

which is a tiny quantity while for low momentum states with $p \simeq 0$

$$\delta E^{\text{low momentum}} \sim \frac{1}{p^2} \gg 1,$$

and wrapping corrections are very large. This is physically intuitive as low momentum states probe larger portions of the space cylinder and thus are more sensitive to wrapping interactions.

It would be very interesting if the computations of [18, 19] could be generalized to generic two magnon states with arbitrary L . If this turns out to be feasible then the

study of the dependence of the anomalous dimensions on the magnon momenta could be an important window into the structure of wrapping effects in $\mathcal{N} = 4$ SYM.

As mentioned above, to compute (2.41) or (2.42) we only need to know the leading g^0 position of the Bethe roots which are simply given by the Heisenberg chain Bethe equations. If $u_1 = -u_2 = u$ they become simply

$$\left(\frac{u+i/2}{u-i/2}\right)^L = \frac{u-(-u)+i}{u-(-u)-i}$$

so that the only effect of the second magnon is to effectively renormalize L to $L-1$. Therefore the momenta $p = 2 \arctan 2u$ will be simply given by $p = -\pi + \frac{2\pi n}{L-1}$. e.g. for the Konishi operator $p = -\frac{\pi}{3}$ and $\delta E = -2268g^8$ which is represented by the mismatch between curves c_4 and c_2 in figure 1.

Another example which illustrates the huge variance of the prefactor as function of the magnon momenta is the 2 magnon state for some large chain with, e.g., $L = 100$. For these states the wrapping corrections range from the smallest values for the largest momentum states with $n = 49$ for which $\delta E \simeq -7.97 \times 10^{-300} g^{200}$ to the highest value for the lowest momentum states with $n = 1$ and corresponding $\delta E = -1.15 \times 10^{64} g^{200}$.

3. Families of long-ranged integrable hamiltonians

In the previous sections, we explored wrapping effects in the Hubbard model, as a controlled toy model closely related to $\mathcal{N} = 4$, see discussion in the introduction. In that model the long range Bethe equations for the magnons are effective equations and the fundamental degrees of freedom are electrons whose interactions are ultra local. In this section, we will explore a completely different toy model which also has an analytic solution. In this model, the fundamental description is given by a long ranged Hamiltonian where, by construction, wrapping interactions are under control. Contrary to the previous model, it does not seem to share many features with the known $\mathcal{N} = 4$ spin chain however it is a nice simple model worth looking at.

The algebraic Bethe ansatz construction is the formalism that diagonalizes transfer matrix operators like (1.5) mentioned in the introduction.⁹ These transfer matrices commute with one another for different values of the spectral parameter.¹⁰ Thus, if we construct a spin chain Hamiltonian H from the transfer matrix (usually, by taking derivatives of its logarithm at a particular point u^*) then, by construction $[H, T(u)] = 0$ and we immediately obtain a huge number of conserved charges and hence quantum integrability.

For example, let us consider the standard $SU(2)$ spin chain transfer matrix

$$\hat{T}(\lambda) \equiv \text{tr}_0 \left(\prod_{j=1}^L \frac{\lambda + i \mathcal{P}_{0j}}{\lambda + i} \right), \tag{3.1}$$

⁹A transfer matrix also appeared in section 2.3.2 although there, the operators being multiplied inside the trace were S -matrices rather than R -matrices.

¹⁰The reason for this is that the operators being multiplied in (1.5) or in (3.1) below obey the Yang-Baxter triangle relation.

where \mathcal{P}_{0j} is the permutation operator between a physical vector space h_j and an auxiliary space h_0 , both isomorphic to \mathbb{C}^2 . The transfer matrix is then an operator acting in the full Hilbert space given by the tensor product of L copies h_1, \dots, h_L associated with the L spin chain sites. The eigenvalues of this $SU(2)$ transfer matrix are given by

$$T(\lambda) = \prod_{j=1}^M \frac{\lambda - u_j - i/2}{\lambda - u_j + i/2} + \left(\frac{\lambda}{\lambda + i} \right)^L \prod_{j=1}^M \frac{\lambda - u_j + 3i/2}{\lambda - u_j + i/2} \quad (3.2)$$

where u_j are denoted by Bethe roots. It is clear from the definition (3.1) that the eigenvalues can not have poles as λ approaches $u_j - i/2$. The cancellation of the corresponding residues in (3.2) is indeed guaranteed by the Bethe equations

$$\left(\frac{u_j + i/2}{u_j - i/2} \right)^L = \prod_{k \neq j}^M \frac{u_j - u_k + i}{u_j - u_k - i} \quad (3.3)$$

which quantize the Bethe roots u_j . Physically, the left hand side of this equation represents the free propagation $e^{ip_j L}$ of the magnon j as it goes around the spin chain while the r.h.s. $\prod_{k \neq j} \mathcal{S}(p_j, p_k)$ describes the scattering of this magnon with all the other magnons. Therefore, the Bethe root u_j and the momentum p_j of the j th magnon are related by

$$u = \frac{1}{2} \cot \frac{p}{2}. \quad (3.4)$$

Having diagonalized \hat{T} we have automatically diagonalized *all* Hamiltonians obtained from this transfer matrix. For example, the Heisenberg Hamiltonian (2.6) can be obtained as $\frac{1}{2i} \frac{d}{d\lambda} \log \hat{T}(\lambda) \Big|_{\lambda=0}$ so that the spectrum is simply

$$E = \frac{1}{2i} \frac{d}{d\lambda} \log T(\lambda) \Big|_{\lambda=0} = \sum_{j=1}^M \frac{1/2}{u_j^2 + 1/4} = \sum_{j=1}^M 2 \sin^2 \frac{p_j}{2}.$$

By considering more derivatives of $\log \hat{T}$ at $\lambda = 0$ we generate other local Hamiltonians with longer range. To obtain the spectrum of these Hamiltonians we simply act with more derivatives on the logarithm of the eigenvalue (3.2). In particular, if we take more than L derivatives, wrapping interactions, where the range of the Hamiltonian is bigger than the size of the spin chain, will appear. Remarkably, for all such models the Bethe equations are just (3.3) since they diagonalize the transfer matrix!

A particularly interesting hamiltonian is

$$\hat{H}(g) = \frac{1}{4i} \log \frac{\hat{T}(g^2)}{\hat{T}(0)} + h.c. \quad (3.5)$$

If we think of g^2 as being an expansion parameter then we have an infinite range Hamiltonian where, at each order g^{2n} in perturbation theory, the interaction range is n . In the notations of [60] we have

$$\hat{H}(g) = \frac{g^2}{2} \sum_j \mathcal{H}_{j,j+1} + \frac{ig^4}{4} \sum_j [\mathcal{H}_{j,j+1}, \mathcal{H}_{j+1,j+2}] + \dots \quad (3.6)$$

where $\mathcal{H}_{j,j+1} = 1 - \mathcal{P}_{j,j+1}$. The spectrum of this Hamiltonian is then given by (3.5) where we simply replace the operators $\hat{T}(\cdot)$ by the corresponding eigenvalues $T(\cdot)$ to get

$$\begin{aligned}
 E &= \frac{1}{4i} \sum_{j=1}^M \log \left(\frac{u_j + \frac{i}{2} - g^2 u_j - \frac{i}{2}}{u_j - \frac{i}{2} - g^2 u_j + \frac{i}{2}} \right) \\
 &\quad + \frac{1}{4i} \log \left[1 + \left(\frac{g^2}{g^2 + i} \right)^L \prod_{j=1}^M \frac{u_j - \frac{3i}{2} - g^2 u_j - \frac{i}{2} - g^2}{u_j + \frac{i}{2} - g^2 u_j + \frac{i}{2} - g^2} \right] + c.c.. \quad (3.7)
 \end{aligned}$$

The first term comes from the first term in (3.2). It gives a contribution to the energy of the form $\sum \epsilon(p_j)$, that is a sum of the dispersion relations of M individual magnons interacting through (3.3). The dispersion relation, when written in terms of p , yields

$$\begin{aligned}
 \epsilon(p) &= \frac{1}{2i} \log \left(\frac{1 - 2g^2 e^{-ip/2} \sin \frac{p}{2}}{1 - 2g^2 e^{+ip/2} \sin \frac{p}{2}} \right) \\
 &= 2g^2 \sin^2 \frac{p}{2} + 2g^4 \sin p \sin^2 \frac{p}{2} + \dots + \frac{(2g^2)^n}{n} \sin \frac{np}{2} \sin^n \frac{p}{2} + \dots \quad (3.8)
 \end{aligned}$$

The second term in (3.7), which comes from the second term in (3.2), is identically zero up to order g^{2L} , precisely when wrapping interactions appear! Thus, at order g^{2L} the energy is given by a sum of dispersion relations of the form (3.8) plus this wrapping term, which entangles all M magnons and is not writable as a sum of individual magnon energies. In terms of the magnon momentum we have

$$E = \sum_{j=1}^M \epsilon(p_j) + g^{2L} \frac{1}{4i} \left[i^{-L} \prod_{j=1}^M (2e^{-2ip_j} - e^{-ip_j}) - c.c. \right] + \mathcal{O}(g^{2L+1}). \quad (3.9)$$

For example, for 1 and 2 magnons we get respectively

$$E(p) = \epsilon(p) + \frac{g^{2L}}{2} \left(\sin \left(p + \frac{L\pi}{2} \right) - 2 \sin \left(2p + \frac{L\pi}{2} \right) \right) + \mathcal{O}(g^{2L+2}), \quad (3.10)$$

and

$$E(p, k) = \epsilon(p) + \epsilon(k) + g^{2L} \left(s(p) + s(k) - \frac{s(0)}{2} - 2s(p+k) \right) + \mathcal{O}(g^{2L+2}), \quad (3.11)$$

where $s(x) \equiv \sin(p+k+L\pi/2+x)$. In particular, it is clear that the correction to the energy of the two magnon state is not of the form $\delta\epsilon(p) + \delta\epsilon(k)$. This was expected since at order g^{2L} the interaction range covers the entire chain and the notion of asymptotic region where one can safely measure the dispersion relation of each magnon is destroyed [9, 59].

When we compare the type of corrections (3.10) and (3.11) we get from this long range model compared with those we got within the Hubbard model (2.39) and (2.42) we see that while in the latter the coefficient of the order g^{2L} wrapping correction to the energy exhibited a strong L dependence, from factors like $\cos^L(p/2)$, in the former this coefficient is L independent and generically of order 1. If this feature is generic then the

weak or strong L dependence of the g^{2L} prefactors to the corrections of 2 magnons states in $\mathcal{N} = 4$ super Yang-Mills could be a sign, respectively, of a *fundamental* description in terms of a long ranged exact integrable Hamiltonian or of a local description in terms of an hidden level of particles in the Beisert-Staudacher nested Bethe ansatz equations. In this context, the perturbative computation of [18] and [19], if generalized to a more general setup with 2 magnons and generic L could provide very useful hints about the nature of the fundamental integrable structure of $\mathcal{N} = 4$ super Yang-Mills.

3.1 Generalizations and transcendentality

In the previous section we saw that we could easily generate (long-ranged) integrable Hamiltonians by considering

$$\hat{H} = \frac{1}{4i} \sum_{n=1}^{\infty} a_n \frac{g^{2n}}{n!} \frac{d^n}{d\lambda^n} \log \hat{T}(\lambda) \Big|_{\lambda=0} + h.c. \tag{3.12}$$

The spectrum of such Hamiltonians is immediately given by this expression with the operator $\hat{T}(\lambda)$ replaced by the corresponding eigenvalue (3.2). At order g^{2n} these Hamiltonians are local with interactions of range n . If we truncate the expansion at a given order m by setting $a_{n>m} = 0$, then for chains of length larger than m we have no wrapping interactions and the energy is simply given by a sum of dispersion relations $\sum_j \epsilon(p_j)$, with¹¹

$$\epsilon(u) = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{a_n g^{2n}}{n} \left(\frac{1}{(u - i/2)^n} - \frac{1}{(u + i/2)^n} \right) \tag{3.13}$$

If, on the other hand, we consider an infinite sum or, alternatively, if we truncate the expansion in g^2 at an order $m > L$, the spectrum (starting at wrapping order g^{2L}) will no longer be a sum of individual dispersion relations. In particular, precisely at order g^{2L} we obtain

$$E = \sum_{j=1}^M \epsilon(u_j) + \frac{g^{2L}}{4i} \left[\frac{a_L}{i^L} e^{-iP} \prod_{j=1}^M \frac{u_j - 3i/2}{u_j + i/2} - c.c. \right] + \mathcal{O}(g^{2L+2}) \tag{3.14}$$

where the second term takes into account the wrapping interactions. It is interesting to notice that for all these families of long ranged Hamiltonians the expression for the wrapping interactions is quite simple and absolutely universal. The example (3.5) we considered corresponds to $a_n = 1$ but as we see any choice of a_n will lead to a solvable problem. A particularly funny example would be

$$\hat{H} = \frac{1}{2i} \sum_{n=1}^{\infty} C_n \frac{g^{2n}}{(2n-1)!} \frac{d^{2n}}{d\lambda^{2n}} \log \hat{T}(\lambda) \Big|_{\lambda=0} + h.c.,$$

with C_n the Catalan numbers, for which we find the curious expression

$$\epsilon(u) = \frac{g}{i} \left(\frac{1}{X^+(u)} - \frac{1}{X^-(u)} \right), \quad X^{\pm}(u) \equiv \frac{u \pm \frac{i}{2} + \sqrt{(u \pm \frac{i}{2})^2 - 4g^2}}{2g}$$

¹¹If the a_n are not real — which from the definition 3.12 is a perfectly reasonable possibility — we should take the real part of this expression.

for the dispersion relation. As a function of the Bethe roots this is precisely the dispersion relation appearing in the BDS equations [4] and even in the full AdS/CFT Bethe equations [8]. However, unfortunately, as a function of the magnon momenta this is not the same as (2.10) because the relation between u and p in our toy models is always of the form (3.4) instead of (2.12). In other words, although the desired dispersion relation can easily be obtained, the Bethe roots are always quantized via the usual Heisenberg spin chain Bethe equations.

We can generalize this construction of long ranged Hamiltonians for other symmetry groups as well. For example, for $SU(N)$ the transfer matrix in the fundamental representation takes exactly the same form (3.1) as for the $SU(2)$ chains except that the operators being multiplied now live in $h_j \otimes h_0$ where both h_0 and h_j are copies of \mathbb{C}^N . Thus, we can still consider Hamiltonians of the form (3.5) with a g^2 perturbative expansion (3.6). As before, they will be long ranged Hamiltonians where the Hamiltonian range at order g^{2n} is n . The only thing that changes is the expression for the eigenvalue (3.2). For an $SU(N)$ spin chain we have $N - 1$ types of roots and K_n Bethe roots of the type $n = 1, \dots, N - 1$. The transfer matrix eigenvalue reads¹²

$$T_{SU(N)}(\lambda) = \prod_{j=1}^{K_1} \frac{\lambda - u_j^{(1)} - i/2}{\lambda - u_j^{(1)} + i/2} + \left(\frac{\lambda}{i + \lambda}\right)^L \sum_{n=1}^{N-1} \prod_{j=1}^{K_n} \frac{\lambda - u_j^{(n)} + \frac{n+2}{2}i}{\lambda - u_j^{(n)} + \frac{n}{2}i} \prod_{j=1}^{K_{n+1}} \frac{\lambda - u_j^{(n+1)} + \frac{n-1}{2}i}{\lambda - u_j^{(n+1)} + \frac{n+1}{2}i}$$

We see again that when we compute the first few local charges expanding $\log T(u)$ around $u = 0$ only the first term gives a non-vanishing contribution. As before we can consider Hamiltonians of the form (3.12). The dispersion relation for the $SU(N)$ magnons is exactly the same as for $SU(2)$ while the leading wrapping correction to the spectrum can be computed as before to yield the generalization of (3.14),

$$E = \sum_{j=1}^{K_1} \epsilon(u_j^{(1)}) + \frac{g^{2L}}{4i} \left[\frac{a_L}{i^L} e^{-iP} \sum_{n=1}^{N-1} \prod_{j=1}^{K_n} \frac{u_j^{(n)} - \frac{n+2}{2}i}{u_j^{(n)} - \frac{n}{2}i} \prod_{j=1}^{K_{n+1}} \frac{u_j^{(n+1)} - \frac{n-1}{2}i}{u_j^{(n+1)} - \frac{n+1}{2}i} - c.c. \right] + \mathcal{O}(g^{2L+2}).$$

where again $P = \frac{1}{i} \sum_j \log \frac{u_j^{(1)} + i/2}{u_j^{(1)} - i/2}$ is the state total momentum.

All these considerations can be trivially generalized both to non-compact spin chains and to supersymmetric ones. For example, for the $SL(2)$ spin chain we have [61, 62]

$$T_{sl(2)}(\lambda) = \prod_{j=1}^M \frac{\lambda - u_j - i/2}{\lambda - u_j + i/2} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{in + \lambda}\right)^L \prod_{j=1}^M \frac{(\lambda - u_j - i/2)^2}{(\lambda - u_j + \frac{2n-1}{2}i)(\lambda - u_j + \frac{2n+1}{2}i)} \tag{3.15}$$

¹²The $SU(N)$ Bethe equations can be immediately obtained by canceling the apparent λ poles in this expression.

with the Bethe equations, following from canceling the poles in this expression, reading

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^M \frac{u_j - u_k - i}{u_j - u_k + i} \tag{3.16}$$

which differ from (3.3) by a simple sign in the r.h.s. Again, by expanding the log of the transfer matrix around $\lambda = 0$ we see that only the first term contributes until the L 'th derivative is taken. Thus if we consider an Hamiltonian of the form (3.12) we will have, up to order g^{2L} , the energy as a sum of the same dispersion relations (3.13). In particular if the constants a_n are algebraic numbers then so will be $\sum_j \epsilon(u_j)$ when truncated to order g^{2L} because clearly the solutions to (3.16) are also algebraic (complex) numbers.

However, precisely at order g^{2L} the second term in (3.15) starts contributing and we find

$$E = \sum_{j=1}^M \epsilon(u_j) + \frac{g^{2L}}{4i} \left[\frac{a_L}{i^L} e^{-iP} \sum_{n=1}^{\infty} \frac{1}{n^L} \prod_{j=1}^M \frac{(u_j + i/2)^2}{(u_j - \frac{2n-1}{2}i)(u_j - \frac{2n+1}{2}i)} - c.c. \right]. \tag{3.17}$$

This new wrapping term differs from the one computed for the compact groups $SU(N)$ by the fact that it is given by an infinite sum of terms. Thus even if u_j and a_n are perfectly algebraic numbers the energy of this state will only be algebraic up to order g^{2L} , when this infinite sum will give a transcendental contribution!

Let us consider a few examples. We chose $a_L = 1$ for simplicity. For $L = 4$ the one magnon state with momentum $2\pi/4$ will be corrected to

$$E = \epsilon(p) - g^6 (1 - \zeta(3)) + \mathcal{O}(g^8)$$

while for example for a two magnon state with $L = 5$ and momenta¹³ $p_1 = -p_2 = p = 2\pi/6$ we get

$$E = \epsilon(p) + \epsilon(-p) + \frac{g^{10}}{4} (1 - 2\zeta(3) + 2\zeta(5)) + \mathcal{O}(g^{12})$$

In table 1 in the introduction we listed a couple of additional examples.

Although, this model is not immediately related to the (non-compact sector of) AdS/CFT Bethe equations which are much more complicated than (3.16), it is still interesting to see that transcendentality very naturally appears due to the non-compact nature of the transfer matrix. In particular, if an extra level of hidden degrees of freedom is to be discovered then the transcendental numbers present in the dressing factor could be an important hint. A more fundamental $PSU(2, 2|4)$ symmetric transfer matrix in the field strength representation would also be given by an infinite sum of terms since the representation is infinite dimensional. It would be spectacular if a relatively simple extended transfer matrix with some extra degrees of freedom included and only simple algebraic

¹³For two magnon with opposite momenta the $sl(2)$ equations are trivially solved exactly as explained in the end of section 2.3.3 for the $su(2)$ chain. For the non-compact chain the effect of the second magnon is simply to renormalize $L \rightarrow L + 1$ instead of $L \rightarrow L - 1$ as we had for the $SU(2)$ chain. Thus we obtain $p_1 = -p_2 = \frac{2\pi n}{L+1}$ for the two magnon state with opposite symmetric momenta.

expressions could lead to the intricate structure of the full dressing factor where transcendental numbers abound. Probably the correct place to try to reverse engineer and find this extra level of hidden particles is the transfer matrix rather than the Bethe equations.

Acknowledgments

We would like to thank N. Gromov, V. Kazakov, A. Kozak, S. Schafer-Nameki, J. Polchinski, P. Ribeiro, and K. Zarembo for several interesting discussions and insightful comments. JP is funded by the FCT fellowship SFRH/BPD/34052/2006. PV is funded by the Fundação para a Ciência e Tecnologia fellowship SFRH/BD/17959/2004/0WA9. PV would like to thank KITP and its several members for the warm hospitality and for partially funding this work. PV would like to thank Caltech Institute of Technology for warm hospitality. This research was supported in part by the National Science Foundation under Grant No. NSF PHY05-51164.

A. Wave function of the Hubbard magnon

In this section we shall study states made out of two fundamental particles o and \uparrow ,

$$|\Psi\rangle = \sum_{n,n'} \psi(n, n') \left(a_{n,\uparrow}^\dagger a_{n',o}^\dagger \right) | \uparrow \dots \uparrow \rangle.$$

Acting on this state with the Hamiltonian (2.1) we can find the form of the wave function $\psi(n, n')$ so that the state is an eigenstate,

$$H|\Psi\rangle = E|\Psi\rangle.$$

This gives the equation

$$E\psi(n, n') = -t \left[e^{i\phi_\uparrow} \psi(n+1, n') + e^{-i\phi_\uparrow} \psi(n-1, n') \right. \\ \left. + e^{i\phi_o} \psi(n, n'+1) + e^{-i\phi_o} \psi(n, n'-1) \right] \quad (\text{A.1})$$

$$(E+U)\psi(n, n) = -t \left[e^{i\phi_\uparrow} \psi(n+1, n) + e^{-i\phi_\uparrow} \psi(n-1, n) \right. \\ \left. + e^{i\phi_o} \psi(n, n+1) + e^{-i\phi_o} \psi(n, n-1) \right] \quad (\text{A.2})$$

with $n' \neq n$. A plane wave superposition

$$\psi(n, n') = e^{-i\phi_\uparrow n - i\phi_o n'} \left(A e^{iqn+iq'n'} + B e^{iq'n+iqn'} \right), \quad n < n' \quad (\text{A.3})$$

$$\psi(n, n') = e^{-i\phi_\uparrow n - i\phi_o n'} \left(C e^{iqn+iq'n'} + D e^{iq'n+iqn'} \right), \quad n > n' \quad (\text{A.4})$$

solves the first equation and yields

$$E = -2t \cos(q) - 2t \cos(q').$$

Continuity of the wave function at $n = n'$ gives

$$A + B = C + D$$

and the second equation reduces to

$$A(2g \sin(q) - 2g \sin(q') - i) = C(2g \sin(q) - 2g \sin(q')) + iB. \quad (\text{A.5})$$

In an infinitely large volume we might look for bound states with exponentially decaying wave functions. These are only possible if two of the exponentials in (A.3) and (A.4) disappear. Parametrizing the momenta as $q = p/2 - i\beta$ and $q' = p/2 + i\beta$ we see that we need $B = C = 0$ and $A = D \neq 0$. Then equation (A.5) fixes

$$\sinh \beta = \sinh \beta_\infty \equiv -\frac{1}{4g \cos \frac{p}{2}} \quad (\text{A.6})$$

in which case

$$\psi(n, n') = e^{-i\phi_\uparrow n - i\phi_o n'} e^{i\frac{p}{2}(n+n')} e^{-\beta|n-n'|}$$

and the energy (A.5) reads

$$E = \epsilon_\infty(p) \equiv -4t \cos \frac{p}{2} \cosh \beta_\infty = -U \sqrt{1 + 16g^2 \cos^2 \frac{p}{2}}.$$

Comparing with the energy (A.5) of two particles with momentum $p/2$ we conclude that the magnon has a relative binding energy

$$\cosh \beta - 1.$$

On the other hand, at finite volume we impose periodicity

$$\psi(n + L, n') = \psi(n, n'), \quad n < n' < n + L$$

and

$$\psi(n, n' + L) = \psi(n, n'), \quad n' < n < n' + L$$

of the wave function to obtain

$$C e^{i(q-\phi_\uparrow)L} = A, \quad D e^{i(q'-\phi_\uparrow)L} = B$$

and

$$A e^{i(q'-\phi_o)L} = C, \quad B e^{i(q-\phi_o)L} = D$$

This immediately gives total momentum quantization

$$e^{i(q+q'-\phi_\uparrow-\phi_o)L} = 1$$

and the relation

$$\frac{B}{C} = \frac{1 - e^{i(q-\phi_\uparrow)L}}{1 - e^{i(q-\phi_o)L}}$$

From (A.5) we obtain

$$e^{i(q-\phi_\uparrow)L} = \frac{2g \sin(q) - 2g \sin(q') + i \frac{1 - e^{i(q-\phi_\uparrow)L}}{1 - e^{i(q-\phi_o)L}}}{2g \sin(q) - 2g \sin(q') - i}$$

which can be rewritten as

$$e^{i(q-\phi_o)L} = \frac{g(\sin(q) - \sin(q'))(1 + e^{i(\phi_\uparrow - \phi_o)L})}{2g \sin(q) - 2g \sin(q') - i} + \frac{\sqrt{g^2(\sin(q) - \sin(q'))^2(1 - e^{i(\phi_\uparrow - \phi_o)L})^2 - e^{i(\phi_\uparrow - \phi_o)L}}}{2g \sin(q) - 2g \sin(q') - i}$$

This is precisely what one obtains from the Lieb-Wu equations (2.4) for the Bethe roots q, q', u eliminating the auxiliary variable u using its Bethe equation. Moreover, in the case of identical twists this becomes the simple S-matrix

$$e^{i(q-\phi_o)L} = \frac{2g \sin(q) - 2g \sin(q') + i}{2g \sin(q) - 2g \sin(q') - i}$$

If again we look for solutions in the form $q = p/2 - i\beta$ and $q' = p/2 + i\beta$ so that

$$E = \epsilon(p) = -4t \cos \frac{p}{2} \cosh \beta,$$

we obtain

$$e^{i(p-\phi_o-\phi_\uparrow)L} = 1 \tag{A.7}$$

and

$$4g \cos \frac{p}{2} \sinh \beta = -\frac{\sinh(\beta L)}{\cosh(\beta L) - \cos L \left(\frac{p}{2} - \phi_o\right)}.$$

Notice that the result is symmetric under the exchange of twists since (A.7) implies

$$\cos L \left(\frac{p}{2} - \phi_o\right) = \cos L \left(\frac{p}{2} - \phi_\uparrow\right) = \cos \frac{L}{2} (\phi_\uparrow - \phi_o).$$

In the case of equal twists the expression reduces to

$$4g \cos \frac{p}{2} \sinh \beta = -\frac{\sinh(\beta L)}{\cosh(\beta L) \pm 1},$$

and for BDS twists obeying $e^{iL(\phi_o-\phi_\uparrow)} = -1$ gives

$$4g \cos \frac{p}{2} \sinh \beta = -\tanh(\beta L).$$

In any case, for large L the r.h.s. can be replaced by -1 and we recover the infinite volume result (A.6).

B. Dressed BDS BAE with twists

This appendix complements the computations and results of sections (2.3.1) and (2.3.3). As we see from (2.34), (2.36) and (2.32) a crucial quantity we need to compute the leading wrapping corrections is Δ_n defined in (2.31). To compute Δ_n we focus on the equation for

a single constituent of the Takahashi triplet, say on equation (2.23) for q_n . Expanding this equation using (2.31) and (2.33) yields

$$\Delta_n \simeq \Delta_n^{(0)} \equiv i e^{i(\frac{p_n}{2} + i\beta_n - \phi_o)L} \prod_{m \neq n}^M \frac{u_m^\infty - u_n^\infty + i}{u_m^\infty - u_n^\infty} = \mathcal{O}(e^{-\beta_n L}), \quad (\text{B.1})$$

to leading order. However, as seen from the above mentioned equations (2.34), (2.36) and (2.32) what we really need is the imaginary part of Δ_n . Due to the corrected BDS equations (2.34) the imaginary part of $\Delta_n^{(0)}$ is of order $e^{-\beta_n L}$ for generic twists and of order $e^{-3\beta_n L}$ when (2.5). Indeed, the imaginary part of $\Delta_n^{(0)}$ can be simplified using the BDS corrected equations (2.34) to give

$$\Im(\Delta_n^{(0)}) \simeq -i \Delta_n^{(0)} e^{i(\phi_o - \phi_\uparrow)\frac{L}{2}} \cos \frac{L}{2}(\phi_o - \phi_\uparrow) \quad (\text{B.2})$$

which indeed vanishes when (2.5). Thus, in this case we need to expand the equations for q_n further,

$$\Delta_n \simeq \Delta_n^{(0)} - i \left(\left(\Delta_n^{(0)} \right)^2 + \sum_{m \neq n}^M \frac{\Delta_n^{(0)} \Delta_m^{(0)}}{(u_m^\infty - u_n^\infty)(u_m^\infty - u_n^\infty + i)} \right). \quad (\text{B.3})$$

To find the corrected BDS equations we multiply the equations for the Takahashi triplet to get

$$e^{i(p_n - \phi_\uparrow - \phi_o)L} \prod_{m \neq n} \frac{u_n^\infty - u_m^\infty - i}{u_n^\infty - u_m^\infty + i} = \prod_{m=1}^M \frac{u_n^\infty - u_m^\infty - i}{u_n^\infty - u_m^\infty + i} \frac{u_n - u_m - i}{u_n - u_m + i} \frac{r_{m,n} r_{m,n+M}}{r_{n,m} r_{n,m+M}} \quad (\text{B.4})$$

where

$$r_{n,m} \equiv \frac{u_n - 2g \sin q_m - i/2}{u_n - 2g \sin q_m + i/2}. \quad (\text{B.5})$$

Notice that so far no approximation whatsoever was done and (B.4) is an exact relation. Expanding now the r.h.s. of (B.4) using (2.31) and (2.33) we find the corrected (twisted) BDS equations (2.34)

Again, when the twists are chosen as in (2.5) we need to be more careful and expand our expressions further. By expanding the product of the equations for the Takahashi triplet to order $e^{-2\beta L}$ we get the same (2.34) with

$$\phi_{nm} = -\frac{\Im(\Delta_n)}{(u_n - u_m)^2 + 1} \left(\frac{1}{u_n} \tan^2 \frac{p_n}{2} + \frac{2}{u_n - u_m} \right) + \frac{2(u_n - u_m)\Delta_n \Delta_m}{((u_n - u_m)^2 + 1)^2} - (n \leftrightarrow m) \quad (\text{B.6})$$

instead of (2.35).

Finally, in (2.36) the energy is corrected because the momenta change (the second term) and because the functional dependence on the momenta changes (the first term). For completeness let us mention why the correction due to the change in functional dependence is so simple. This is simply because

$$-4t \cos \frac{p_n}{2} \cosh(\beta_\infty(p_n) + \delta\beta_n) = -U \left(-4g \cos \frac{p_n^\infty}{2} \sinh \beta_\infty(p_n^\infty) \right) \delta\beta_n = -U \delta\beta_n \quad (\text{B.7})$$

since $-4g \cos \frac{p_n^\infty}{2} \sinh \beta_\infty(p_n^\infty) = 1$.

C. Luscher with magnon-magnon S-matrix at strong coupling

In this appendix we expand the results of [30]¹⁴ concerning the BDS Luscher term to the next order in $1/g$. The computation in question is the μ -term

$$\begin{aligned} \delta\epsilon(p) &= \frac{1}{2} \text{Res}_{k=k^*} e^{iLk} [\epsilon'(p) - \epsilon'(k)] \frac{u(p) - u(k) + i}{u(p) - u(k) - i} + c.c. \\ &= e^{iLk^*} [\epsilon'(p) - \epsilon'(k^*)] \frac{i}{-u'(k^*)} + c.c. \end{aligned}$$

where

$$u(p) = \frac{1}{2} \tan \frac{p}{2} \sqrt{1 + 16g^2 \cos^2 \frac{p}{2}}$$

and k^* is defined by the pole condition

$$u(p) - u(k^*) + i = 0.$$

For large g , this gives

$$k^* = \pi + \frac{i}{2g} \sec \frac{p}{2} + \frac{1}{4g^2} \tan \frac{p}{2} \sec^2 \frac{p}{2} + \frac{i}{384g^3} (23 \cos p - 49) \sec^5 \frac{p}{2} + \dots$$

which yields

$$\delta\epsilon(p) = \Re \left(-\frac{U}{g} \sec \frac{p}{2} + \frac{iU}{g^2} \tan \frac{p}{2} \sec^2 \frac{p}{2} + \dots \right) \exp -\frac{L}{2g} \sec \frac{p}{2} \left(1 - \frac{i}{2g} \tan \frac{p}{2} \sec \frac{p}{2} + \dots \right)$$

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¹⁴The notation in this paper differs from ours by $p_{\text{there}} = p_{\text{here}} + \pi$, $g_{\text{there}} = \sqrt{2}g_{\text{here}}$. Moreover, we have an extra factor of U multiplying the magnon dispersion relation. This shift in p introduces an annoying $(-1)^L$ factor. In order to avoid carrying it around we consider L even in this appendix.

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